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CALCULATION OF THE RADIAL DISTRIBUTION OF THE DENSITY DEPENDENT PROPERTIES IN AN AXISYMMETRIC GAS STREAM

By

M. T. Dooley and W. K. McGregor
RTF, ARO, Inc.

May 1961

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ABSTRACT

An analytical method of determining the radial distribution of the density dependent properties in an axisymmetric gas stream is presented. This method is used to invert the integrated lateral intensity distribution of properties which are normally obtained with experimental techniques to the desired local radial variation.

The equation of lateral distribution is derived, and the analytical solution for the radial distribution equation is shown. A numerical solution ideally suited to high speed digital computers is presented. A particular application, determination of the radial distribution of spectral radiant emission from a plasma jet, is described; a discussion of the shapes of the measured and inverted distribution is also included.

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INTRODUCTION

The density dependent properties of axisymmetric gas streams arrange themselves in iso-circles about the axis of the gas stream and so may be considered as functions of the radius. Any of the density dependent properties, such as light intensity, whether emitted or absorbed, can readily be measured laterally by placing the measuring device at right angles to the gas stream. The resulting measurement is then an integration across all the iso-circles encountered by a line passing from the measuring device and perpendicular to the stream axis. These lateral measurements provide a distribution F(x) which is a function of the distance from the stream centerline and not a function of the radius r. Therefore, since a radial distribution is of major interest, some method must be devised to invert the lateral measurements to functions of the radius.

The method of lateral to radial inversion consists, essentially, of deriving an equation for the lateral measurement as a function of the unknown property I(r), the radius r, and the parameter x, where x is the horizontal distance from the stream centerline. The derived equation can then be inverted analytically to achieve a solution for the unknown property I(r) as a function of the horizontal distance x with r as a parameter. The inverted solution is then an integral equation of such form that an analytical solution is not readily available. Therefore, a numerical solution is resorted to for the integration.

The problem of radial inversion has been considered previously (Refs. 1 and 2) and methods presented for its solution. However, these methods tend to avoid the lateral to radial inversion integral or to replace the inverted equation by a summation.

In this report a complete derivation of the equation expressing the lateral distribution is presented. The solution of the lateral distribution equation for the radial distribution is also presented, along with a method of numerical solution for the radial distribution that lends itself to methods applicable to a digital computer. In the numerical solution, the replacement of the inverted expression by a numerical series, as reported by other investigators, is avoided. Instead, an appropriate change of variable is made so that the integral can be replaced by the area under a curve determined from experimental data. This area represents the solution and can be found very accurately by many different methods (Ref. 3).

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The method of radial inversion presented in this report is quite general and is applicable to any density dependent property of an axisymmetric gas stream. The lateral measurements can be made with schlieren optical systems, interferometers, or shadowgraphs, as well as with spectrographic techniques.

For purposes of illustration the lateral measurement of spectral light intensity of an axisymmetric plasma stream is inverted to give the intensity as a function of the radius. The data are taken from a simple d-c arc-excited plasma jet using spectographic techniques (Ref. 4).

DERIVATION OF THE LATERAL EQUATION

The lateral measurement of a property e. g., light energy $F(x)$ in an axisymmetric plasma stream, can be represented schematically as in Fig. 1 where $F(x)$ represents the column of light energy of unit depth, dx width, and length from $-\infty$ to $+\infty$. Therefore, since $F(x)$ is the sum of all the elements $I(r)\,dy$, $F(x)$ can be expressed as

$$F(x) = \int_{-\infty}^{+\infty} I(r)\,dy \tag{1}$$

From symmetry

$$F(x) = 2\int_{0}^{+\infty} I(r)\,dy$$

and from geometry

$$y = (r^2 - x^2)^{1/2}$$

$$dy = \frac{r\,dr}{(r^2 - x^2)^{1/2}}$$

Now, substitution in Eq. 1 yields

$$F(x) = 2\int_{x}^{R} \frac{I(r)\,r\,dr}{(r^2 - x^2)^{1/2}} \tag{2}$$

assuming that there are no external contributions of light energy outside the region $\pm$ R. Equation (2) is a form of Abel's Integral Equation (Ref. 5) and can be readily inverted analytically for $I(r)$. The inversion for $I(r)$ is shown in detail in Appendix A and yields

$$I(r) = \frac{1}{\pi}\int_{r}^{R} \frac{F'(x)\,dx}{(x^2 - r^2)^{1/2}} \tag{3}$$

METHOD OF EVALUATING THE RADIAL EQUATION

A solution to the radial equation, Eq. (3), is not immediately obvious because of the nonavailability of an analytical expression for $F'(x)$. Also, many values of r must be used to determine a true radial distribution. Moreover it is desirable to use a method of solution for Eq. (3) that is readily adaptable to digital computers.

Consider Eq. (3)

$$I(r) = \frac{1}{\pi} \int_r^R \frac{F'(x)\, dx}{(x^2 - r^2)^{1/2}}$$

Put in the form

$$I(r) = \frac{1}{\pi} \int_r^R \frac{F'(x)}{x} \left(\frac{x\, dx}{(x^2 - r^2)^{1/2}} \right)$$

and with the substitution

$$u = (x^2 - r^2)^{1/2}$$

$$du = \frac{x\, dx}{(x^2 - r^2)^{1/2}}$$

Eq. (3) becomes

$$I(r) = \frac{1}{\pi} \int_0^R \frac{F'(x)}{x}\, du$$

which can be readily integrated by plotting $F'(x)/x$ vs u and measuring the area under the resulting curve.

The lack of an analytical expression for $F(x)$ precludes the use of any method other than a numerical method for determining $F'(x)/x$. However, if care is used in selecting a numerical method the error encountered will be negligible (Ref. 3).

Consider the lateral distribution data $F(x_o)$, $F(x_1) \ldots F(x_n)$; then the value of the derivative at each x can be calculated by

$$F'(x_n) = \frac{1}{\Delta x} \left[\frac{\Delta F(x_{-1}) + \Delta F(x_{+1})}{2} \right] \tag{4}$$

where

$$\Delta F(x_{-1}) = F(x_n) - F(x_{n-1})$$

$$\Delta F(x_{+1}) = F(x_n) - F(x_{n+1})$$

and

$$\Delta x = x_{n+1} - x_n$$

if x is given in equal increments.

To calculate I(r), first consider

$$\pi I(r) = A = \frac{u_1}{2}(y_1 + y_0) + \frac{u_2 - u_1}{2}(y_2 + y_1) + \cdots$$

$$\frac{u_n - u_{n-1}}{2}(y_n + y_{n-1}) \tag{5}$$

This equation is derived from Fig. 2 where the shaded areas are considered to be triangles. The simplicity of the formula is necessary since a more accurate formula (e. g. the Gauss formula for unequal intervals - Ref. 3) would present calculations that are nearly prohibitive in their complexity. This simplicity of Eq. 5 makes it quite applicable to digital computers; the fact that the intervals on the abcissa do not have to be equal makes it quite versatile. The only requirement is that the values of Δu be kept small.

EXAMPLE PROBLEM

To illustrate the method of radial inversion presented in this report, an analytical function that lends itself to rapid solution is used to generate sample data. Thus, a check is provided on the method of solution, and any error becomes apparent.

To generate data the expression $F(x) = e^{-x^2}$ is used. When this expression is substituted in Eq. (3)

$$I(r) = \frac{1}{\pi}\int_r^R \frac{-2xe^{-x^2}}{(x^2 - r^2)^{1/2}}\,dx = \frac{1}{\sqrt{\pi}}\, e^{-r^2}\, \mathrm{Erf}\,(R^2 - r^2)^{1/2} \tag{6}$$

The substitution of four values of r into Eq. (6) results in the values given in column two of Table 1. The third column represents values computed by the method presented in this report.

TABLE 1

Radius, r/R	Exact Value, I(r)	Value of Numerical Solution, $I_n(r)$	Error, e, (%)
0.00	0.56	0.55	1.85
0.34	0.21	0.20	1.85
0.67	0.01	0.01	0.00
1.00	0.00	0.00	0.00

To compute the values in column three, the numerical formula for $F'(x)$ (Eq. 4) is first applied. The the computation of $F'(x)/x$ for each x gives the values of the ordinates for the area curve. If the equation $u = (x^2 - r^2)^{1/2}$ is applied for a particular r, each x yields the values on the abcissa of the area curve (Fig. 2). The area under the resulting curve is determined using Eq. (5) and, after it is divided by π, gives I(r).

Column four shows the error in percent for the various solutions. The reader is referred to Fig. 3 for a graph of the complete solution. The superimposed dotted curve is the solution by the method presented in this report. The greatest deviation of the dotted curve from the actual solution is of the order of 1. 85 percent.

RESULTS AND DISCUSSION

EXPLANATION OF CURVE SHAPES

The principal application of the method presented in this report has been the radial inversion of the lateral integrated spectral intensity of an Argon plasma jet. The plasma stream was an axisymmetric jet about 7 mm in diameter and about 25-mm long (Ref. 4). The measurements of intensity were made using a direct recording spectrometer set at 4195Å. At this particular wave length only continuous radiation is present.

Several sets of experimental data obtained at different planes of measurement are shown in Fig. 4. After the radial inversion was performed on these data, the result was the I(r) vs r plots of Fig. 5. It should be noted that, as the plane of measurement is moved, there is a definite shift in the location of the peak value of the I(r) curves. Therefore, since the data curves change shape radically as this measurement plane is moved, it is assumed that the movement of the peak of the I(r) curve is directly related to the shape of the data curve. Figure 6 shows the peak values of each I(r) curve as a function of the area under the corresponding data curve.

The sample curves of Fig. 7 were generated to illustrate the shift of peak values, and the radial inversion was performed on each to obtain the solutions of Fig. 8. Note how the peaks of the curves uniformly shift toward r = 0 as the area under the corresponding data curve decreases. To explain this shift of peak value, consider Fig. 9. Here a cross section of the gas stream is represented, with superimposed profiles of the lateral intensity for various planes of measurement (curves 1, 2, 3). If the section of measurement of width Δx (shaded area) moves to the right, the section rapidly decreases in length. However, if at the same time the lateral profile remains flat (curve number 1 on Fig. 9), then the peak occurs off-center. This is obvious if one considers that although the area element of intensity has decreased in size, the lateral profile of measurement has remained relatively constant, an indication that the intensity of some radial point [say point (A) on Fig. 9] is greater than a point at the geometrical center.

CONCLUDING REMARKS

The method of calculating the radial distribution of the density dependent properties in an axisymmetric gas stream presented in this report is quite general. It has application to any lateral distribution of data that is obtained from external observations. Thus the method verifies that set forth by Brinkman (page 401 of Ref. 6) and is in close agreement with the results obtained by other investigators (Refs. 1 and 2).

To illustrate another use for the method of radial inversion, consider, for example, Fig. 10, which is a photograph of a plasma exhausting to low pressure with a probe inserted in the flame. (Calculations using shock wave angle measurements indicated a Mach number on the order of 2.8.) The structure of the shock wave cannot be observed laterally since it is axisymmetric. However, if the negative of the photograph is analyzed on a densitometer, a lateral distribution of the density dependent properties is obtained (Fig. 11a). When the method described in this report is applied, the data yield the radial result of Fig. 11b. The boundaries of the shock wave are clearly defined as a function of the radius. Therefore, it would be quite simple to completely define the boundaries of an axisymmetric shock wave using the method of radial inversion.

Obviously the method is also applicable to data taken by schlieren optical systems. The only stipulations that must be imposed on the method are that the gas stream be axisymmetric and that the desired property be density dependent.

The method of radial inversion described in this report has been programmed for the Royal McBee LGP-30 digital computer. It requires about two hours to solve the problem for inputs of about 30 values of radius in a floating decimal point interpretive routine. A machine language program would reduce computer running time by a factor of at least one-third.

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APPENDIX A

SOLUTION FOR THE RADIAL EQUATION

The lateral equation of the density dependent properties in an axisymmetric gas stream is a form of the well known Abel Integral Equation. Methods of solution of this equation for the radial distribution are given in many references, (e.g. Ref. 7). However, for purposes of simplicity the following algebraic solution is presented.

If both sides of the lateral equation (Eq. 2)

$$F(x) = 2\int_x^R \frac{I(r)\,r\,dr}{(r^2 - x^2)^{1/2}} \tag{A1}$$

are multiplied by $\dfrac{x\,dx}{(x^2 - t^2)^{1/2}}$ integrated over the limits t to R the result is

$$\int_t^R \frac{x\,F(x)}{(x^2 - t^2)^{1/2}}\,dx = \int_t^R \left[\int_x^R \frac{2r\,I(r)\,dr}{(r^2 - x^2)^{1/2}}\right] \frac{x\,dx}{(x^2 - t^2)^{1/2}} \tag{A2}$$

At this point the order of integration of the right hand member of Eq. A2 can be changed if the limits of integration are modified accordingly. To modify the limits of integration the bounds of integration are first constructed graphically as in Fig. A1.

The outer limits of the right hand member of Eq. (A1) are from x = t to x = R and are represented by the dotted lines on Fig. A1. The inner limits are from r = x to r = R, and these are represented by solid lines. The bounds of integration are defined by the shaded area. Inverting the order of integration of Eq. A2 and integrating over the same shaded area,

$$\int_t^R \frac{x\,F(x)}{(x^2 - t^2)^{1/2}}\,dx = \int_t^R \left[\int_t^r \frac{x\,dx}{(r^2 - x^2)^{1/2}(x^2 - t^2)^{1/2}}\right] 2r\,I(r)\,dr \tag{A3}$$

Where the modified limits are from x = t (the dotted line) to the solid line x = r. The limits for the integral with respect to r are from r = t to r = R. This obviously is the same shaded area that was previously used so the value of the right hand member of Eq. (A2) is unchanged.

Consider now the inner integral of the right hand member of Eq. A3

$$\int_t^r \frac{x\,dx}{(r^2 - x^2)^{1/2}(x^2 - t^2)^{1/2}}$$

Completing the denominator results in

$$2\int_t^r \frac{x\,dx}{\left[(r^2 - t^2)^2 - [2x^2 - (r^2 + t^2)]^2\right]^{1/2}}$$

which, when the substitution

$$\phi = 2x^2 - (r^2 + t^2)$$

$$d\phi = 4x\,dx$$

is made, becomes

$$\frac{1}{2}\int_{(t^2-r^2)}^{(r^2-t^2)} \frac{d\phi}{\left[(r^2 - t^2)^2 - \phi^2\right]^{1/2}} = \frac{1}{2}\sin^{-1}\frac{\phi}{(r^2 - t^2)}\Bigg|_{(t^2-r^2)}^{(r^2-t^2)} = \pm\frac{\pi}{2}$$

Equation (A3) now becomes

$$\int_t^R \frac{x\,F(x)}{(x^2 - t^2)^{1/2}}\,dx = \pm\,\pi\int_t^R r\,I(r)\,dr \tag{A4}$$

Differentiating both sides with respect to t and then rearranging yields

$$t\,I(t) = \pm\frac{1}{\pi}\frac{d}{dt}\int_t^R \frac{x\,F(x)}{(x^2 - t^2)^{1/2}}\,dx \tag{A5}$$

The differentiation of the right hand member of Eq. (A5) can now be performed. One method is to first integrate by parts.
Setting

$$u = F(x) \qquad dv = \frac{x\,dx}{(x^2 - t^2)^{1/2}}$$

$$du = F'(x)\,dx \qquad v = (x^2 - t^2)^{1/2}$$

results in

$$t\,I(t) = \frac{1}{\pi}\frac{d}{dt}\left[F(x)\,(x^2 - t^2)^{1/2}\right]\Bigg|_t^R - \int_t^R (x^2 - t^2)^{1/2}\,F'(x)\,dx$$

$$t\,I(t) = \pm\frac{1}{\pi}\frac{d}{dt}\left[-\int_t^R (x^2 - t^2)^{1/2}\,F'(x)\,dx\right]$$

$$t\,I(t) = \pm\frac{1}{\pi}\int_t^R \frac{t\,F'(x)\,dx}{(x^2 - t^2)^{1/2}}$$

and dividing through by t

$$I(t) = \pm\frac{1}{\pi}\int_t^R \frac{F'(x)\,dx}{(x^2 - t^2)^{1/2}}$$

If t is set equal to r, this equation becomes

$$I(r) = \frac{1}{\pi}\int_r^R \frac{F'(x)}{(x^2 - r^2)^{1/2}}\,dx \tag{A6}$$

which is the radial equation expressing the intensity as a function of radius. The positive sign is chosen in that a negative value of intensity is absurd.

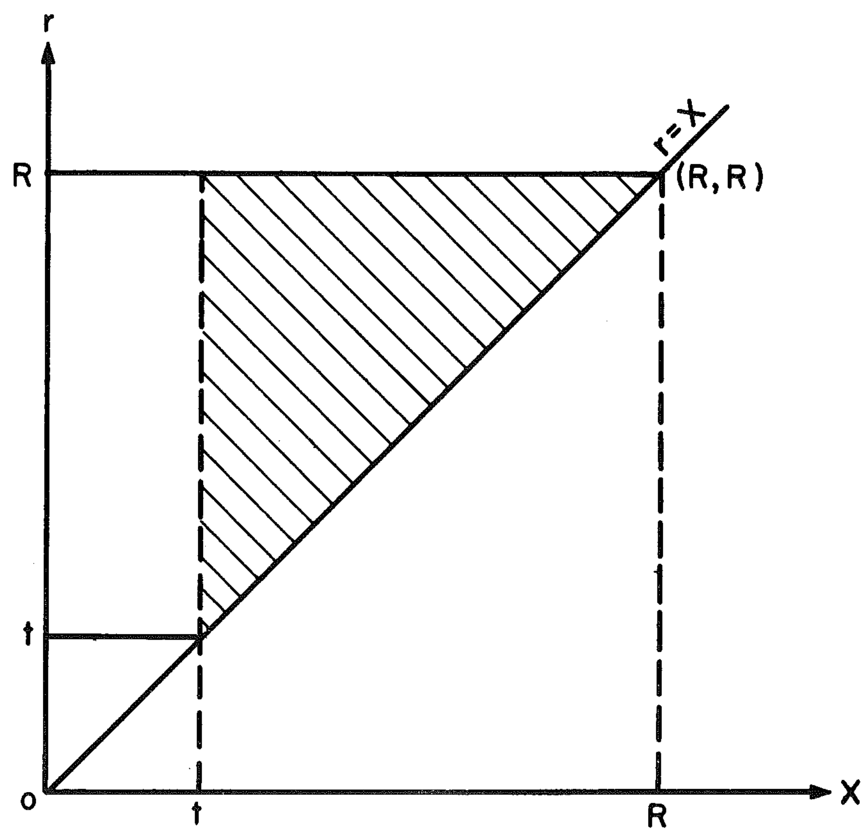


Fig. 1A Graph of the x, r Plane for Bounds of Integration

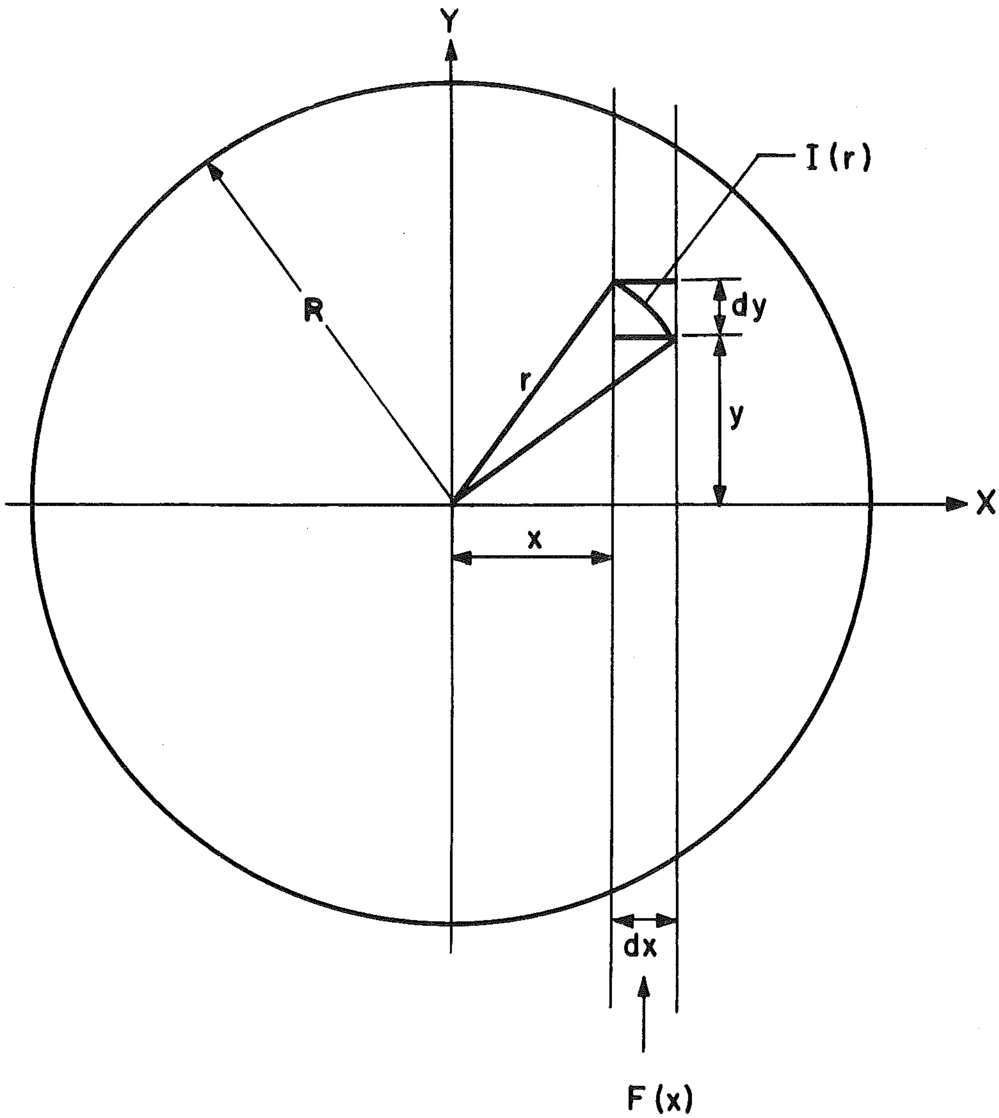


Fig. 1 Cross Section of Axisymmetric Gas Stream

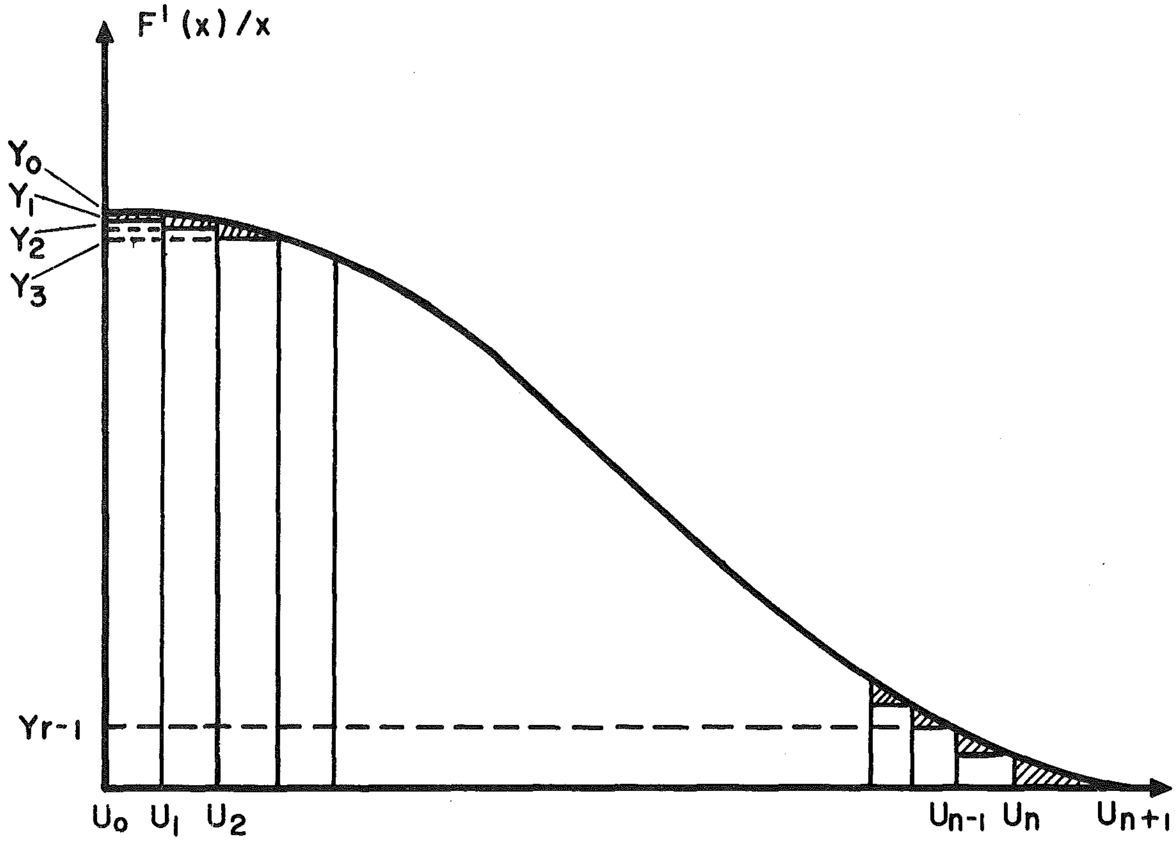


Fig. 2 Plot of F′(x)/x versus u for Determining Area under the Curve

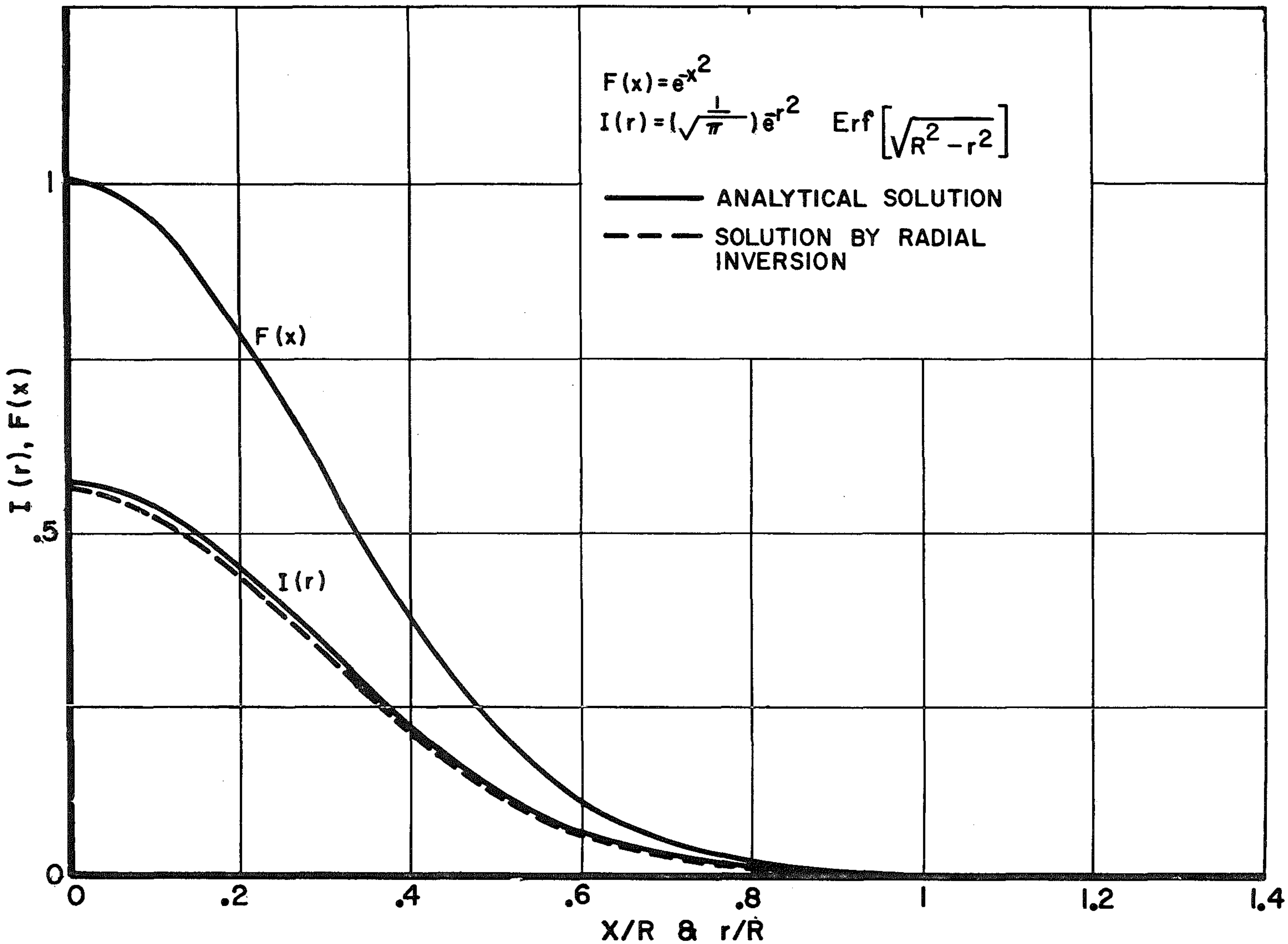


Fig. 3 Data from Sample Problem

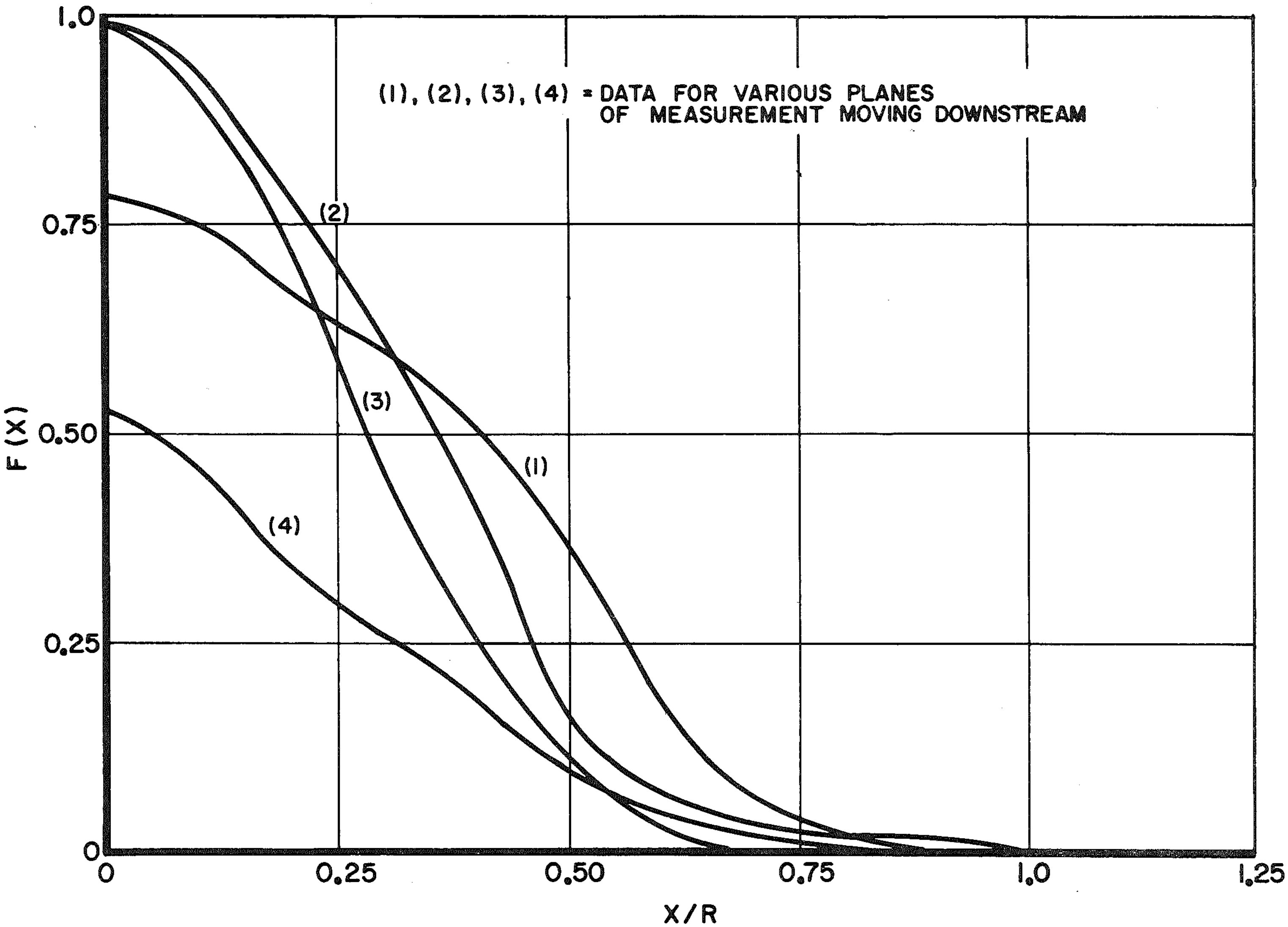


Fig. 4 Lateral Intensity as a Function of Distance from Stream Centerline

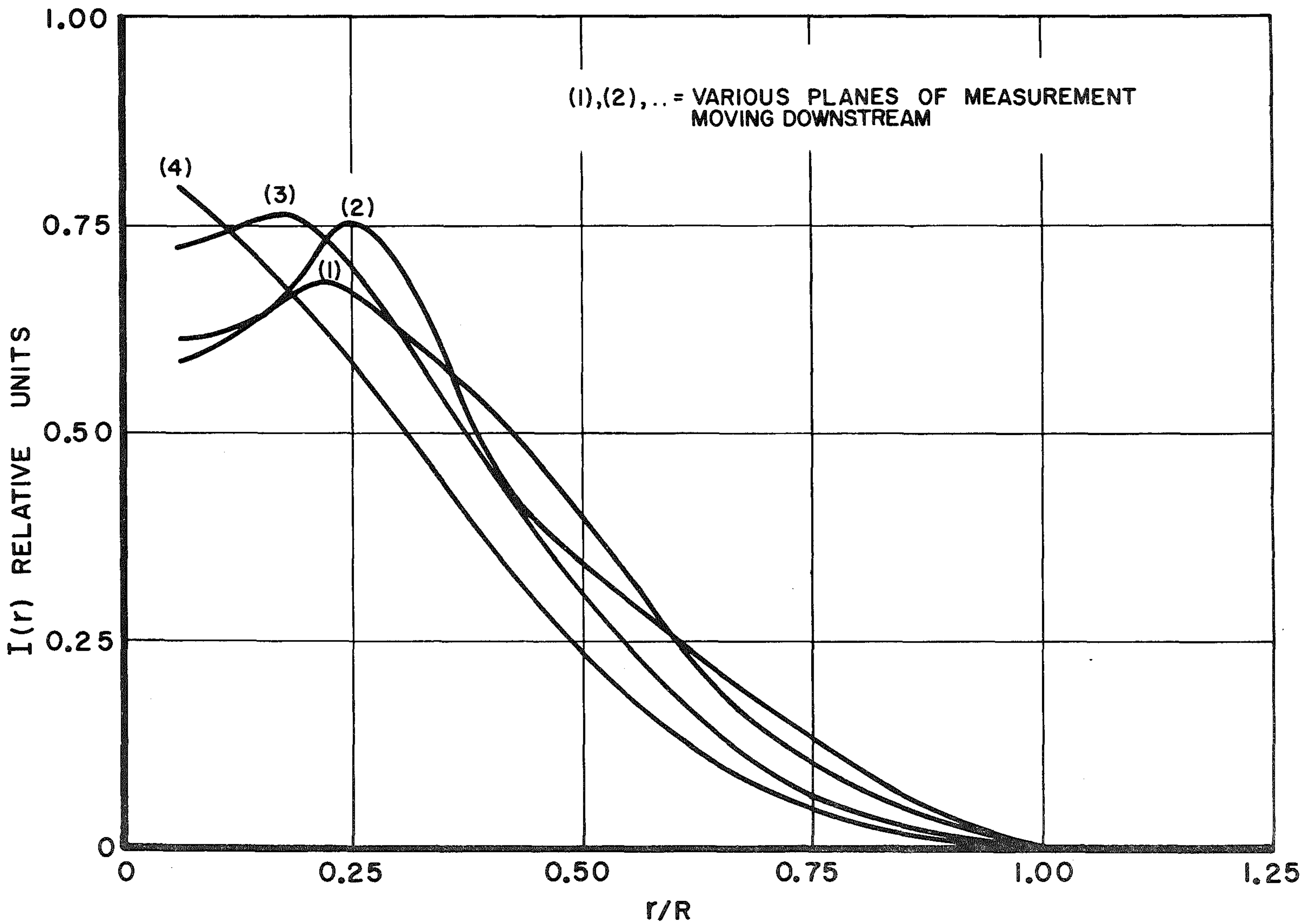


Fig. 5 Radial Intensity versus Radial Distance for Various Planes of Measurement

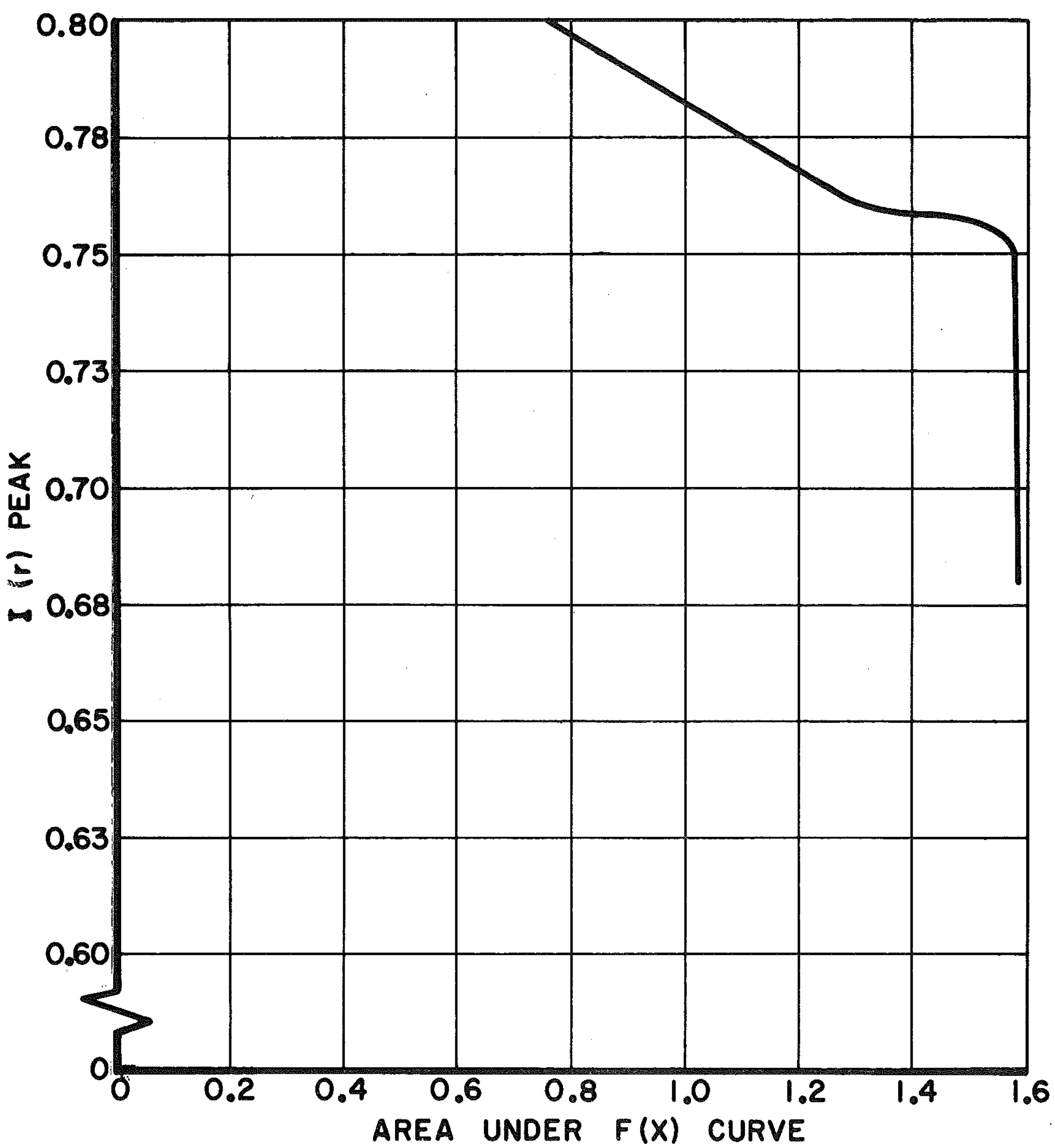


Fig. 6 Relative Peak Intensity as a Function of Area under the Corresponding F (x) Curve

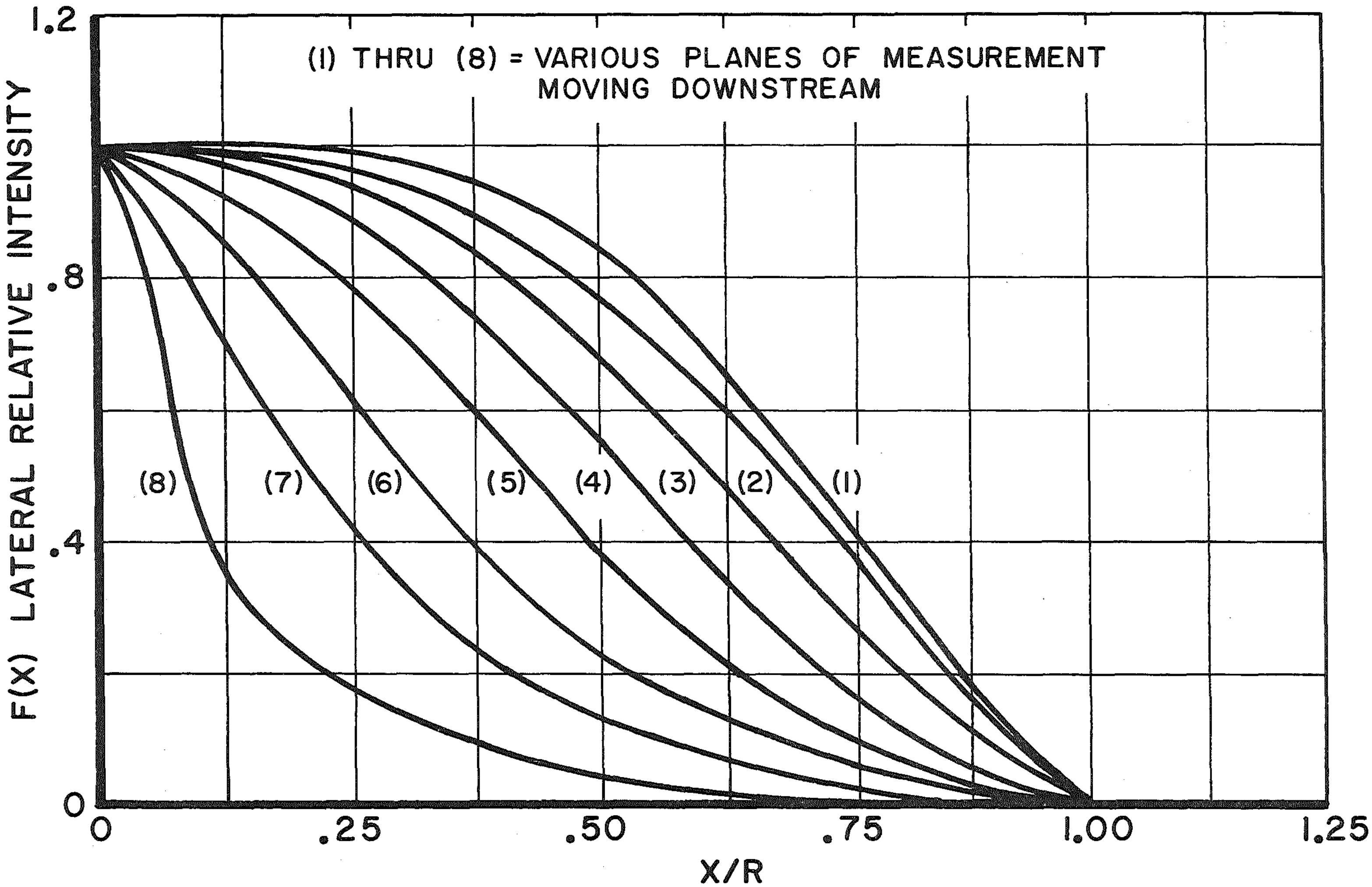


Fig. 7 Sample Lateral Intensity Data

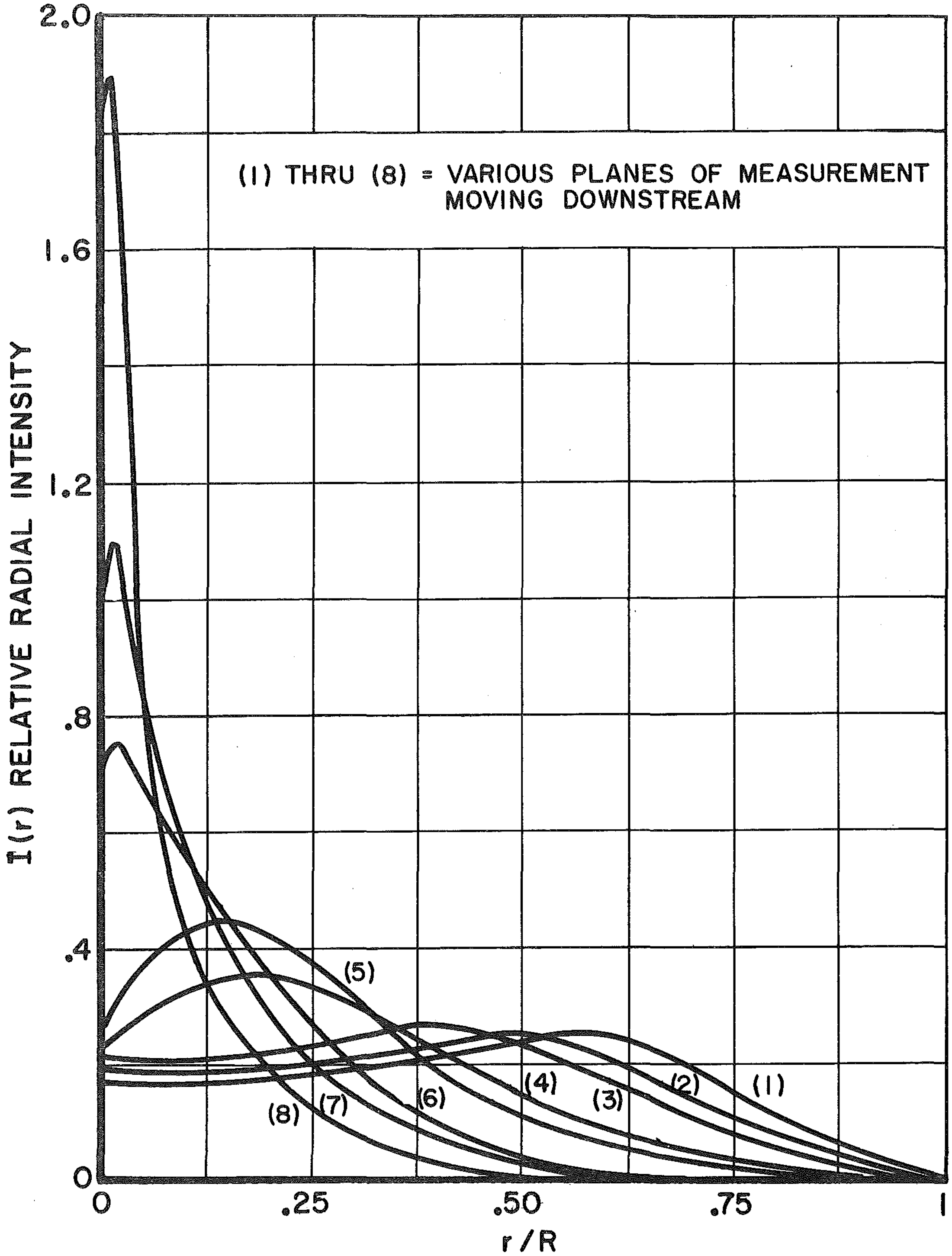


Fig. 8 Sample Radial Intensity Data

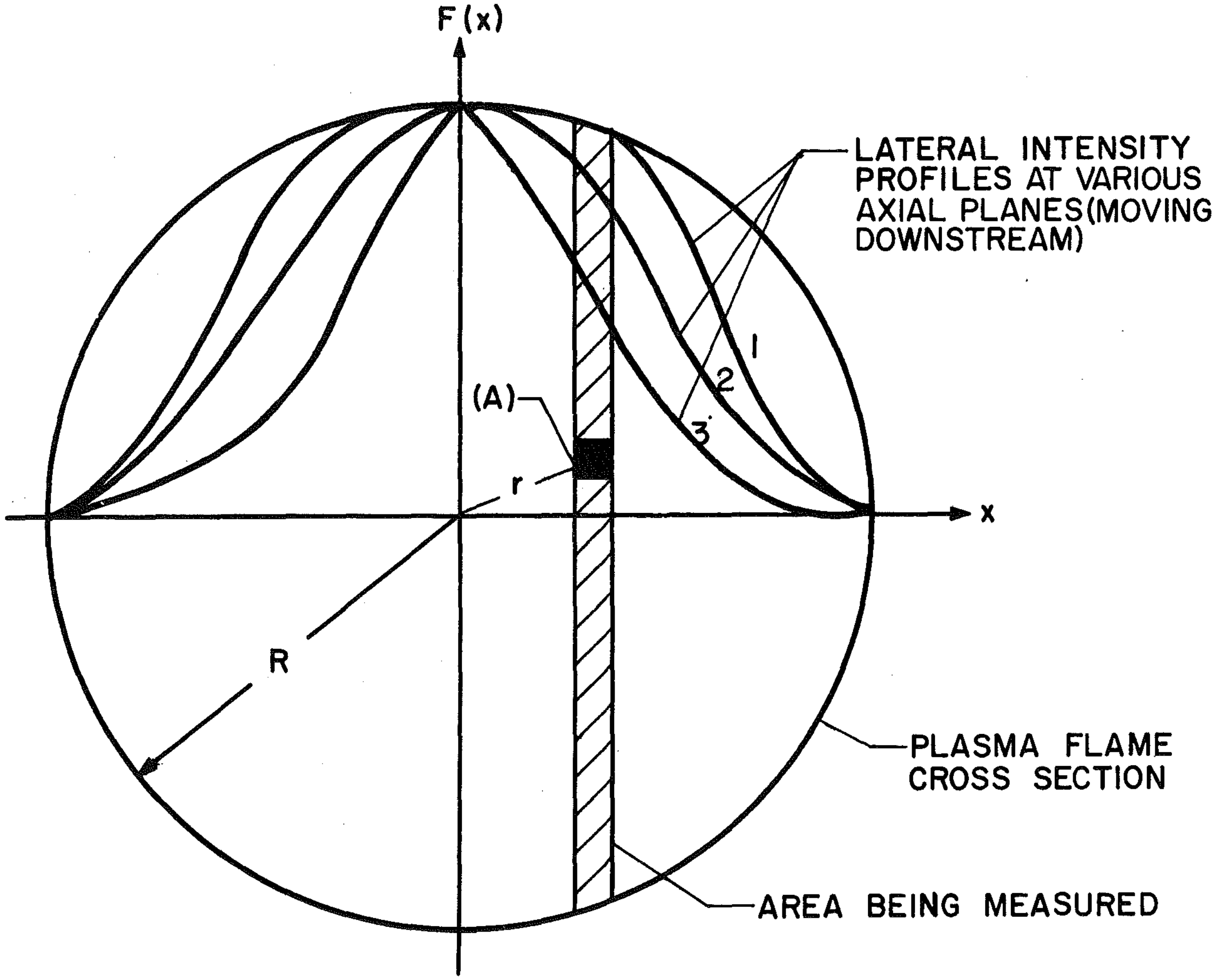


Fig. 9 Superimposed Lateral Intensity Distribution on Cross Section of Plasma Flame

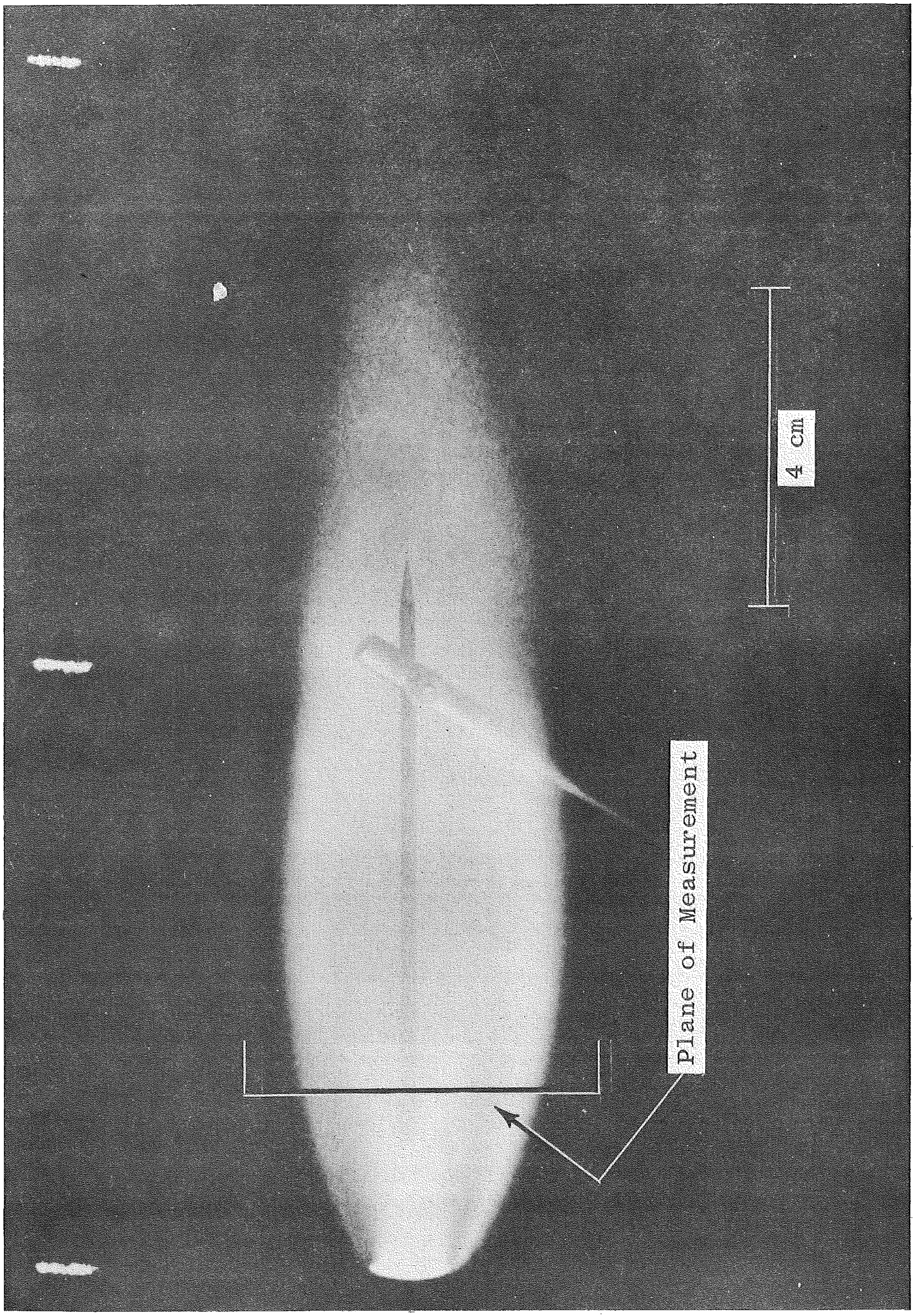
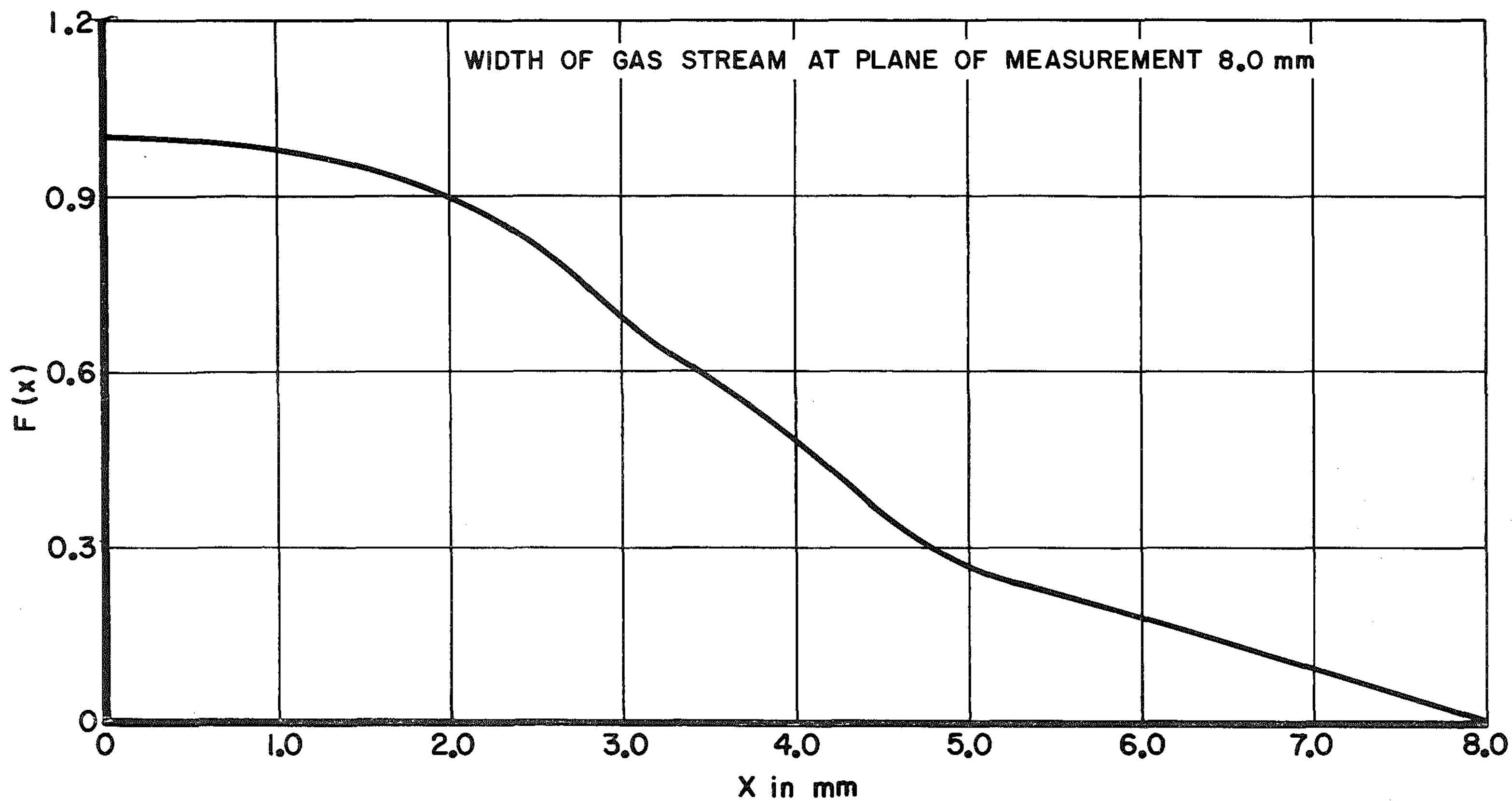
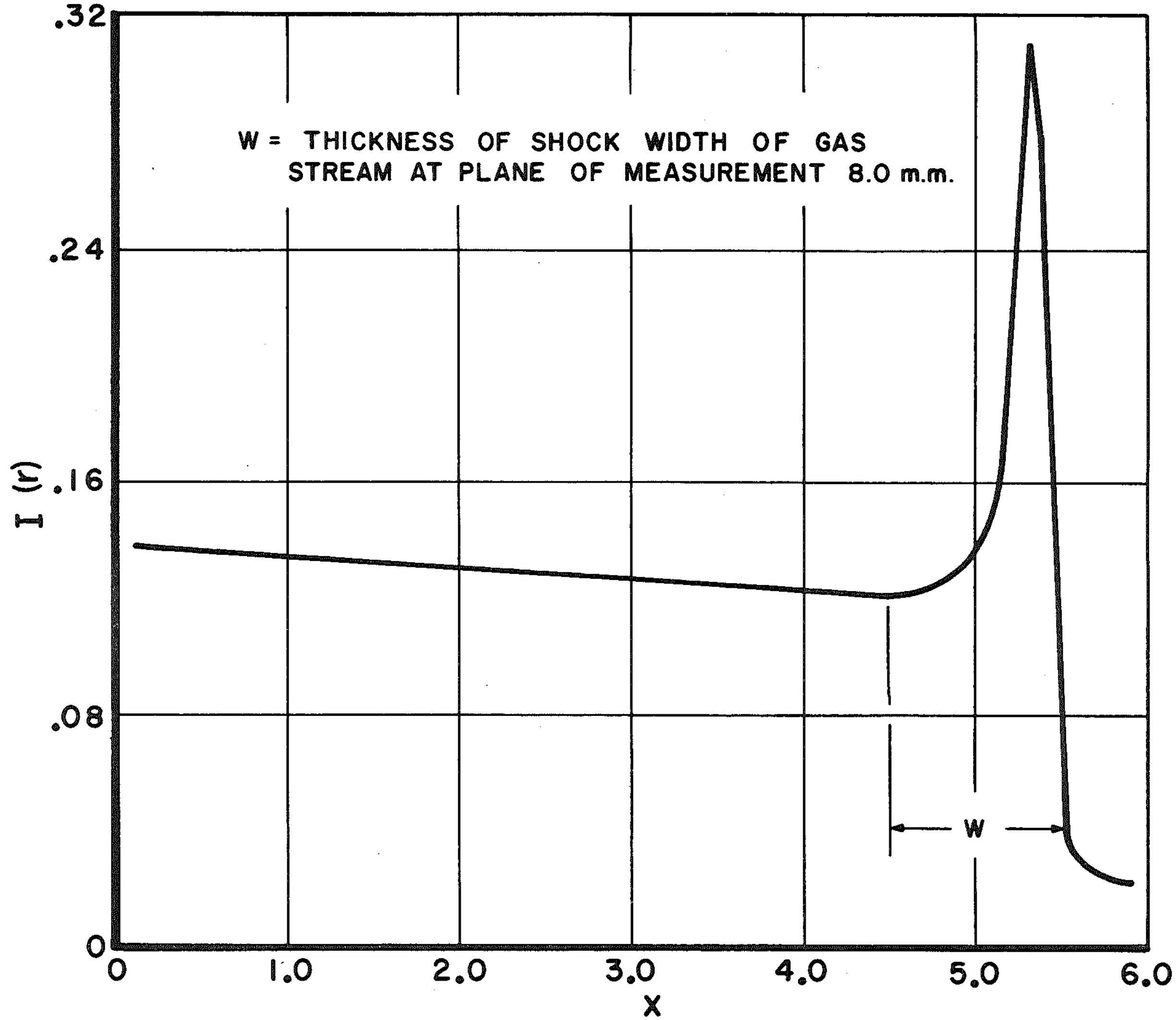


Fig. 10 Photograph of Plasma Exhausting to Low Pressure



a. Lateral Intensity Distribution

Fig. 11 Intensity Distribution of Plasma Exhausting to Low Pressure



b. Radial Intensity Distribution

Fig. 11 Concluded